

Instability of the Fermi sea against surface plasma oscillations

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We derive a generic formalism for studying the energy conversion processes in bounded metals. Using this formalism we show that the Fermi sea of ultra-pure metals could be unstable against surface plasma oscillations, which opens for the latter an intrinsic self-amplification channel. The origin of the instability is clarified and its robustness is revealed. The amplification rate of this channel is analytically evaluated on the basis of energy conservation and the effects of losses are discussed. In particular, the unique role played by the surface in energy conversion is unveiled.

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I. INTRODUCTION

At low temperatures electrons reside in a sphere in the momentum space, known as the Fermi sea, provided they are free and independent¹. Upon turning on their interactions, the Fermi sea can become unstable². A familiar example is superconductivity, where even a tiny short-range attractive force between the electrons could destabilize the Fermi sea, resulting in an exponential growth of the Cooper pairing amplitude³. Superconductivity represents a thermodynamic instability and is therefore a static phenomenon. Here we discuss a different type of instability, which occurs at a finite frequency and is manifested by an exponential while oscillatory increase of the amount of charges accumulated on the surfaces of metals. This instability is caused by surface plasma waves (SPWs) – density undulations of electrons sustained by long-range Coulomb forces and propagating along metal surfaces.

SPWs constitute a ubiquitous entity in optics involving metals^{4–7}. Earnest studies of SPWs begun over half a century ago when R. Ritchie investigated the energy losses of electrons passing through an aluminum foil^{8,9}. A comprehensive understanding was soon accomplished of many fundamental properties of SPWs in the following decade or so¹⁰. Since then studies on SPWs have become largely application oriented and remarkable progresses have been made in a plethora of areas in the past two decades^{11,12}. However, hitherto most existing studies have presumed that the SPWs are underpinned with diffusive electrons^{13,14}. In this work we take a complementary perspective by assuming that the underlying electrons are ballistic. In addition to such fundamental interest in ballistic SPWs, there is also a practical reason. Many SPW-based applications are hampered by energy losses due to electronic collisions^{15–21}. In order to reduce such losses, materials of high quality are being actively sought^{22,23}. In these materials electrons are highly ballistic and a theory of SPWs underpinned with such ballistic electrons should then be of great value in guiding future experiments. Despite the interest, ballistic SPWs have so far received little attention.

Recently²⁴ we studied ballistic SPWs in an ideal yet prototypical system, namely, a semi-infinite metal (SIM) occupying the half space $z \geq 0$ with a geometric surface located at $z = 0$. The metal is described by the jellium model and inter-band transitions are accordingly neglected. We found that SPWs in this system are unstable and can spontaneously amplify in the

collision-less limit. The amplification indicates a growing-up of the electrostatic potential energy of the system. Now that the total energy must be conserved, an increase of potential energy implies a decrease of the kinetic energy stored in the Fermi sea, thereby signifying an instability of the latter.

In the present work, we aim to provide a detailed picture of the energy conversion involved in ballistic SPWs in the SIM. This picture not only complements the equation of motion approach used in our previous study but also furnishes a physically transparent explanation of the instability and amplification scenario. We show that the instability is subsequent to the interplay between ballistic electronic motions and the surface. These motions allow SPWs to draw energy from the electrons when a surface is present. More precisely, we find that the electrical current density $\mathbf{J}(\mathbf{x}, t)$ can be split in two disparate components, which we call $\mathbf{J}_D(\mathbf{x}, t)$ and $\mathbf{J}_B(\mathbf{x}, t)$, respectively. They are discriminated by their relations with the electric field $\mathbf{E}(\mathbf{x}, t)$ generated by the charge density $\rho(\mathbf{x}, t)$ of the system. It turns out that $\mathbf{J}_D(\mathbf{x}, t)$ relates to $\mathbf{E}(\mathbf{x}, t)$ essentially by Drude-Ohm's law, regardless of the value of τ . For this reason, $\mathbf{J}_D(\mathbf{x}, t)$ is called the diffusive component. However, $\mathbf{J}_B(\mathbf{x}, t)$ has no simple relation with $\mathbf{E}(\mathbf{x}, t)$. What is peculiar with $\mathbf{J}_B(\mathbf{x}, t)$ is that, it would totally disappear without the surface and thus represents genuine surface effects. We call it the surface-ballistic component. We calculate the rate of growth of the electrostatic potential energy and equate it with the work done per unit time on the electrical currents by the electric field $\mathbf{E}(\mathbf{x}, t)$ to obtain the self-amplification rate γ_0 . This calculation echoes Dawson evaluating the rate of Landau damping²⁵ but is more subtle due to the presence of a surface. We find that $\mathbf{J}_B(\mathbf{x}, t)$ imparts a net amount of kinetic energy of the electrons to the waves and is responsible for the instability. The present study provides a general framework for discussing the effects of inter-band transitions, radiation and Landau damping on the instability.

In the next section, we introduce a rigorous framework for studying the energy conversion in the presence of a surface. A generic equation of energy balance is established and employed to prove the instability of a SIM. The critical role of the surface in energy conversion, which has so far not been recognized, is unveiled and highlighted. In Sec. III, we prescribe the electronic distribution function, whose structure is analyzed in Sec. IV. We show that this distribution function splits into a diffusive one and a surface-ballistic one, whose definitions are quantitatively established. In Secs. V and VI,

we evaluate the work done by the electric field on the electrons via the diffusive and the surface-ballistic component, respectively. It is shown that the diffusive current is always oriented perpendicular to the local electric field and therefore no instability would take place if only these motions were present. On the contrary, the surface-ballistic current is directed either along the electric field or opposite to it, resulting in a net transfer of energy from the electrons to the field and thence seeing the instability. As such, SPWs can amplify. The amplification rate is calculated in Sec. VII, according to the method outlined in Sec. II. In Sec. VIII, we discuss the result and summarize the paper. Finally, an appendix is provided to take care of some calculations.

II. ENERGY CONVERSION WITH A SURFACE

The system to be studied is a SIM described in our previous work²⁴. In accord with the jellium model, we treat it as a free electron gas embedded in a static background of uniformly distributed positive charges and confined to the half space $z \geq 0$. In equilibrium it is neutral everywhere. Perturbing the system by for example a beam of light leads to a variation in the concentration of electrons and the appearance of a charge density $\rho(\mathbf{x}, t)$. With no regard to the underlying dynamics of the charges, be it classical or quantum mechanical, the equation of continuity must hold, namely, $(\partial_t + \tau^{-1})\rho(\mathbf{x}, t) + \partial_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}, t) = 0$, where $\mathbf{j}(\mathbf{x}, t)$ stands for the electrical current density solely due to the presence of an electric field $\mathbf{E}(\mathbf{x}, t)$. We have included a damping term $-\rho(\mathbf{x}, t)/\tau$ to account for the electrical currents that arise from fast motions like e.g. electronic collisions, which have nothing to do with $\mathbf{E}(\mathbf{x}, t)$ but tend to equilibrate the system. τ denotes the relaxation time, which by definition approaches infinity in the dissipation-less limit. Throughout we write $\mathbf{x} = (x, y, z)$ and reserve $\mathbf{r} = (x, y)$ for planar coordinates.

The effects of a surface are two-fold. Firstly, the surface scatters and redistributes the electrons, an aspect to be examined in the next section. Secondly, the surface prevents any electrons from escaping the metal and $\mathbf{j}(\mathbf{x}, t)$ must identically vanish for $z < 0$. Thus, we write $\mathbf{j}(\mathbf{x}, t) = \Theta(z)\mathbf{J}(\mathbf{x}, t)$, where $\Theta(z)$ denotes the Heaviside step function. With this prescription the equation of continuity becomes

$$(\partial_t + \tau^{-1})\rho(\mathbf{x}, t) + \partial_{\mathbf{x}} \cdot \mathbf{J}(\mathbf{x}, t) = -\delta(z)J_z(\mathbf{r}, z=0, t), \quad (1)$$

Here $\delta(z)$ is the Dirac function peaked at $z = 0$ and $J_z(\mathbf{x}, t)$ denotes the z -component of $\mathbf{J}(\mathbf{x}, t)$. Equation (1) can also be derived other ways²⁶ and serve as the equation of motion for $\rho(\mathbf{x}, t)$ if we express $\mathbf{J}(\mathbf{x}, t)$ as a functional of $\rho(\mathbf{x}, t)$ by means of Maxwell's equations. As discussed previously²⁴, bulk plasma waves, for which $J_z(\mathbf{r}, z=0, t) = 0$, are governed only by the left hand side of this equation, while SPWs are described by solutions with non-vanishing $J_z(\mathbf{r}, z=0, t)$. If $J_z(\mathbf{r}, z=0, t)$ identically vanished, no charges could build up on the surface and $\rho(\mathbf{x}, t)$ could not peak there: SPWs would not exist.

We may prescribe all field quantities in the form of a plane wave propagating along positive x direction with a complex

frequency $\omega = \omega_s + i\gamma$. We write $\rho(\mathbf{x}, t) = \text{Re}[\rho(z)e^{i(kx - \omega t)}]$, $\mathbf{J}(\mathbf{x}, t) = \text{Re}[\mathbf{J}(z)e^{i(kx - \omega t)}]$ and $\mathbf{E}(\mathbf{x}, t) = \text{Re}[\mathbf{E}(z)e^{i(kx - \omega t)}]$, where $k \geq 0$ denotes the wave number and Re takes the real part of a quantity. We also introduce $\rho_q = \int_0^\infty dz \cos(qz)\rho(z)$ and $\rho_s = \rho_{q=0}$, where $q \geq 0$ and ρ_q shall be taken real-valued in this paper. A cut-off q_c may be imposed on q , because $\rho(z)$ can not vary significantly over the mean inter-particle spacing $\sim n^{-1/3}$; otherwise, the jellium model would break down. Thus, $q_c \sim n^{1/3}$.²⁷ Here n denotes the mean concentration of electrons. In terms of the characteristic frequency of the metal, $\omega_p = \sqrt{4\pi ne^2/m}$, with e being the charge and m the mass of an electron, we have $\omega_s \approx \omega_p/\sqrt{2}$. We calculate γ by the principle of energy balance.

The electrostatic potential energy is given by

$$E_p(t) = \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t)\phi(\mathbf{x}, t),$$

where $\phi(\mathbf{x}, t)$ is the electrostatic potential satisfying $\partial_{\mathbf{x}}^2\phi(\mathbf{x}, t) + 4\pi\rho(\mathbf{x}, t) = 0$. One may be tempted to think that the rate of change of $E_p(t)$ can be directly calculated as the negative of the work done per unit time by the electric field $\mathbf{E}(\mathbf{x}, t) = -\partial_{\mathbf{x}}\phi(\mathbf{x}, t)$ on the electrons, which is, however, not true. The reason is because $E_p(t)$ does not count all the potential energy in the system. Specifically, it does not include the surface potential energy, which may be written $E_s(t) = \int d^3\mathbf{x} \rho(\mathbf{x}, t)\phi_s(\mathbf{x})$, where $\phi_s(\mathbf{x})$ denotes the surface potential. For an ideal surface, $\phi_s(\mathbf{x})$ should vanish in the metal but rise to infinity everywhere on the surface so that no electrons can escape the metal.²⁶ In the electrostatic and dissipation-less limit, energy conservation dictates that $\dot{E}_p(t) = -\dot{E}_s(t) - \dot{E}_k(t) = \int d^3\mathbf{x} \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{E}_s(\mathbf{x}) - P^{(1)}(t)$, where $\mathbf{E}_s(\mathbf{x}) = -\partial_{\mathbf{x}}\phi_s(\mathbf{x})$ and

$$P^{(1)}(t) = \int d^3\mathbf{x} \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t)$$

with the dot taking the time derivative. This relation explains why \dot{E}_p is not given by $-P^{(1)}(t)$. The details of $\phi_s(\mathbf{x})$ and $\mathbf{E}_s(\mathbf{x})$, however, can hardly be known and could vary greatly from one sample to another.

Notwithstanding, since the surface effects have been totally incorporated in the continuity equation, we can deduce a complete equation of energy balance from it. To this end, we multiply Eq. (1) by $\phi(\mathbf{x}, t)$ and integrate it over \mathbf{x} . As both $\rho(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$ evolve by the factor $e^{i(kx - \omega t)}$, we have $\int d^3\mathbf{x} \phi(\mathbf{x}, t)\partial_t\rho(\mathbf{x}, t) = \dot{E}_p(t)$. Further, by integration by parts, $\int d^3\mathbf{x} \phi(\mathbf{x}, t)\partial_{\mathbf{x}} \cdot \mathbf{J}(\mathbf{x}, t) = P^{(1)}(t)$. Similarly,

$$P^{(2)}(t) = \int d^3\mathbf{x} \phi(\mathbf{x}, t)\delta(z)J_z(\mathbf{r}, 0, t)$$

comes from the right hand side of Eq. (1) and reflects genuine surface effects. It can be rewritten

$$P^{(2)}(t) = \frac{1}{2} \int d^3\mathbf{x} J_z(\mathbf{r}, 0, t)E_z(\mathbf{x}, t).$$

Piecing everything, we arrive at

$$\left(\frac{2}{\tau} + \partial_t\right)E_p(t) = -P^{(1)}(t) - P^{(2)}(t), \quad (2)$$

which expresses the energy balance in the system.

With $\rho(\mathbf{x}, t) = \rho(z) \cos(kx - \omega_s t) e^{\gamma t}$, the electrostatic potential can be written $\phi(\mathbf{x}, t) = \phi(z) \cos(kx - \omega_s t) e^{\gamma t}$, where $\phi(z) = (2\pi/k) \int dz' e^{-k|z-z'|} \rho(z')$. As for the electric field, we can write

$$\begin{pmatrix} E_x(\mathbf{x}, t) \\ E_z(\mathbf{x}, t) \end{pmatrix} = e^{\gamma t} \begin{pmatrix} E_x(z) \sin(kx - \omega_s t) \\ E_z(z) \cos(kx - \omega_s t) \end{pmatrix}, \quad (3)$$

For later use we also let $\mathbf{E}(z) = (-iE_x(z), E_z(z))$. In terms of ρ_q , we have

$$\begin{pmatrix} E_x(z) \\ E_z(z) \end{pmatrix} = \int_0^\infty \mathcal{D}q \rho_q \begin{pmatrix} 2 \cos(qz) - e^{-kz} \\ 2(q/k) \sin(qz) - e^{-kz} \end{pmatrix}, \quad (4)$$

where $\mathcal{D}q = dq 4k/(k^2 + q^2)$ denotes a measure.

Generally $\rho(z)$ spreads over a layer of thickness $\sim v_F/\omega_s = k_s^{-1}$ within the surface. One can show that $E_x(z) \approx E_z(z) \approx E(z) = 2\pi\rho_s e^{-kz}$ outside the layer, while in the surface layer $E_x(z)$ and $E_z(z)$ are distinctly different and take opposite signs. Nonetheless, this layer makes a negligible contribution, of the order of $\kappa = k/k_s$, to E_p , which becomes clear if we write $E_p(t) = \frac{1}{8\pi} \int d^3\mathbf{x} \mathbf{E}^2(\mathbf{x}, t)$. As an approximation, we may neglect the contribution of this layer and the potential energy per unit area is then given by²⁸ $\mathcal{E}_p(t) = E_p(t)/S \approx (\pi\rho_s^2/4k) e^{2\gamma t}$, where S denotes the surface area. Thus,

$$\dot{\mathcal{E}}_p(t) \approx \gamma (\pi\rho_s^2/2k) e^{2\gamma t}. \quad (5)$$

Upon substituting this expression in Eq. (2), we end up with an equation for γ if $P^{(1)}(t)$ and $P^{(2)}(t)$ are obtained as functions of γ . What remains to be done is to find $\mathbf{J}(\mathbf{x}, t)$ and to use it to calculate $P^{(1,2)}(t)$. We do this in the following sections.

III. ELECTRONIC DISTRIBUTION FUNCTION

We describe the metal by the jellium model and ignore inter-band transitions accordingly. Boltzmann's theory can then be used to study the electrical response of the system. Surface scatters electrons. In principle, such scattering can be handled with a microscopic surface potential $\phi_s(\mathbf{x})$.²⁶ However, this approach is impractical, for $\phi_s(\mathbf{x})$ varies from one sample to another and its detailed knowledge can hardly be obtained precisely. Alternatively, those effects may be dealt with using phenomenological boundary conditions.²⁹⁻³³ This is possible because $\phi_s(\mathbf{x})$ acts only on the surface and in the bulk the electronic distribution function $f(\mathbf{x}, \mathbf{v}, t)$ obtained as solutions to Boltzmann's equation can be written down without explicitly referring to the surface. A few parameters shall

occur in the solutions and their values reflect on surface properties. In the present paper, we will follow this approach to study the electrical responses.

As usual we divide the distribution function in two terms, $f(\mathbf{x}, \mathbf{v}, t) = f_0(\varepsilon(\mathbf{v})) + g(\mathbf{x}, \mathbf{v}, t)$, where $\varepsilon(\mathbf{v}) = \frac{m}{2} \mathbf{v}^2$ is the energy spectrum while $f_0(\varepsilon)$ is the Fermi-Dirac function giving the equilibrium distribution and $g(\mathbf{x}, \mathbf{v}, t)$ denotes the deviation. Let us write $g(\mathbf{x}, \mathbf{v}, t) = \text{Re}[g(\mathbf{v}, z) e^{i(kx - \omega t)}]$. In the regime of linear response, Boltzmann's equation reads

$$\frac{\partial g(\mathbf{v}, z)}{\partial z} + \lambda^{-1} g(\mathbf{v}, z) + e \frac{\mathbf{v} \cdot \mathbf{E}(z)}{v_z} \frac{\partial f_0}{\partial \varepsilon(\mathbf{v})} = 0, \quad (6)$$

where $\lambda = v_z/\tilde{\omega}$ with $\tilde{\omega} = \omega + i/\tau - kv_x$. The general solution is given by

$$g(\mathbf{v}, z) = e^{-\frac{z}{\lambda}} \left(C(\mathbf{v}) - \frac{e \partial_{\mathbf{v}} f_0}{m v_z} \cdot \int_0^\infty dz' e^{\frac{z'}{\lambda}} \mathbf{E}(z') \right),$$

where $C(\mathbf{v})$ is an arbitrary integration constant to be determined by boundary conditions. We require $g(\mathbf{v}, z) = 0$ distant from the surface, i.e. $z \rightarrow \infty$. For electrons moving away from the surface, $v_z > 0$, this condition is automatically fulfilled. For electrons moving toward the surface, $v_z < 0$, we must choose $C(\mathbf{v}) = \frac{e \partial_{\mathbf{v}} f_0}{m v_z} \cdot \int_0^\infty dz' e^{\frac{z'}{\lambda}} \mathbf{E}(z')$, which leads to $g(\mathbf{v}, z) = \frac{e \partial_{\mathbf{v}} f_0}{m v_z} \cdot \int_z^\infty dz' e^{\frac{z'-z}{\lambda}} \mathbf{E}(z')$. To determine $C(\mathbf{v})$ for $v_z > 0$, the boundary condition at $z = 0$ has to be used, which, whoever, depends on surface properties. For perfectly rough surfaces, we may assume that all electrons emerging at $z = 0$ are thermalized and hence $g(\mathbf{v}, 0) = 0$, i.e. $C(\mathbf{v}) = 0$, so that $g(\mathbf{v}, z) = -\frac{e \partial_{\mathbf{v}} f_0}{m v_z} \cdot \int_0^\infty dz' e^{\frac{z'-z}{\lambda}} \mathbf{E}(z')$ for $v_z > 0$. On the other hand, if a fraction p (Fuchs parameter) of the electrons impinging on the surface are bounced back in the absence of $E_z(z)$, i.e. $g(v_x, v_y, v_z > 0, z = 0) = p g(v_x, v_y, -v_z, z = 0)$ for $E_z(z) = 0$, we get $C(\mathbf{v}) = -p \frac{e \partial_{\mathbf{v}} f_0}{m v_z} \cdot \int_0^\infty dz' e^{-\frac{z'+z}{\lambda}} \mathbf{E}(z')$ for $v_z > 0$. By definition p varies from zero to unity.

In line with the above analysis, we may write

$$g(\mathbf{v}, z) = \Theta(v_z) (g_{>}^{(1)}(\mathbf{v}, z; k) + g_{>}^{(2)}(\mathbf{v}, z; k)) + \Theta(-v_z) g_{<}(\mathbf{v}, z; k),$$

where $g_{>}^{(1)}(\mathbf{v}, z)$ counts the electrons emerging away from the surface, $g_{>}^{(2)}(\mathbf{v}, z)$ counts the electrons bounced back by the surface and $g_{<}(\mathbf{v}, z)$ counts all electrons moving toward the surface. Using Eq. (4), these quantities can be cast in the following form

$$\begin{pmatrix} g_{>}^{(1)}(\mathbf{v}, z; k) \\ g_{>}^{(2)}(\mathbf{v}, z; k) \\ g_{<}(\mathbf{v}, z; k) \end{pmatrix} = -\frac{\partial f_0(\varepsilon(\mathbf{v}))}{\partial \varepsilon} \frac{e}{|v_z|} \int_0^\infty \mathcal{D}q \rho_q \begin{pmatrix} g_{>}^{(1)}(\mathbf{v}, z; k, q) \\ g_{>}^{(2)}(\mathbf{v}, z; k, q) \\ g_{<}(\mathbf{v}, z; k, q) \end{pmatrix}, \quad (7)$$

where the integrands are given by

$$g_{>}^{(1)}(\mathbf{v}, z; k, q) = \frac{v_z - i v_x}{k + i \tilde{\omega}/v_z} e^{-kz} + \frac{2 \left(q v_z \frac{q}{k} + \tilde{\omega} \frac{v_x}{v_z} \right)}{(\tilde{\omega}/v_z)^2 - q^2} \cos(qz) + i \frac{2 \left(q v_x + \tilde{\omega} \frac{q}{k} \right)}{(\tilde{\omega}/v_z)^2 - q^2} \sin(qz) + \left(\frac{i v_x - v_z}{k + i \tilde{\omega}/v_z} - \frac{2 \left(\tilde{\omega} \frac{v_x}{v_z} + q v_z \frac{q}{k} \right)}{(\tilde{\omega}/v_z)^2 - q^2} \right) e^{i \frac{\tilde{\omega} z}{v_z}}, \quad (8)$$

Note that $e^{i\frac{\tilde{\omega}z}{v_z}}$ becomes a pure phase factor in the dissipation-less limit for real ω . Additionally,

$$g_{>}^{(2)}(\mathbf{v}, z; k, q) = p \left[\frac{iv_x - v_z}{k - i\tilde{\omega}/v_z} + \frac{2\left(\tilde{\omega}\frac{v_x}{v_z} - qv_z\frac{q}{k}\right)}{(\tilde{\omega}/v_z)^2 - q^2} \right] e^{i\frac{\tilde{\omega}z}{v_z}}, \quad (9)$$

together with

$$g_{<}(\mathbf{v}, z; k, q) = \frac{iv_x - v_z}{k - i\tilde{\omega}/|v_z|} e^{-kz} + \frac{2\left(q|v_z|\frac{q}{k} + \tilde{\omega}\frac{v_x}{|v_z|}\right)}{(\tilde{\omega}/v_z)^2 - q^2} \cos(qz) - i \frac{2\left(qv_x + \tilde{\omega}\frac{q}{k}\right)}{(\tilde{\omega}/v_z)^2 - q^2} \sin(qz). \quad (10)$$

The electrical current density is calculated as $\mathbf{J}(\mathbf{x}, t) = \text{Re}[\mathbf{J}(z)e^{i(kx - \omega t)}]$, where $\mathbf{J}(z) = e(m/2\pi\hbar)^3 \int d^3\mathbf{v} \mathbf{v} g(\mathbf{v}, z)$. However, the charge density is not given by $\tilde{\rho}(\mathbf{x}, t) = e(m/2\pi\hbar)^3 \text{Re}[e^{i(kx - \omega t)} \int d^3\mathbf{v} g(\mathbf{v}, z)]$, which differs from the actual density $\rho(\mathbf{x}, t)$ by a term exactly localized on the surface. Actually, $\mathbf{J}(\mathbf{x}, t)$ and $\tilde{\rho}(\mathbf{x}, t)$ obeys the equation $\partial_t \tilde{\rho}(\mathbf{x}, t) + \partial_{\mathbf{x}} \cdot \mathbf{J}(\mathbf{x}, t) = 0$, of which no SPWs are admitted, rather than the continuity equation (1).

IV. DIFFUSIVE VERSUS SURFACE-BALLISTIC

The distribution function provided in Eqs. (7) - (10) possesses a notable structure, which becomes evident if we write $g(\mathbf{v}, z) = g_D(\mathbf{v}, z) + g_B(\mathbf{v}, z)$, where $g_D(\mathbf{v}, z)$ collects all terms containing one of the three functions e^{-kz} , $\cos(qz)$ and $\sin(qz)$ while $g_B(\mathbf{v}, z)$ includes only the terms with $e^{i\tilde{\omega}z/v_z}$. Obviously, $g_D(\mathbf{v}, z) = g_{>,D}^{(1)}(\mathbf{v}, z) + g_{<}(\mathbf{v}, z)$ and $g_B(\mathbf{v}, z) = g_{>,B}^{(1)}(\mathbf{v}, z) + g_{>}^{(2)}(\mathbf{v}, z)$, where $g_{>}^{(1)}(\mathbf{v}, z)$ has been split by the same token. $g_D(\mathbf{v}, z)$ does not depend on p . It goes without saying that both $g(\mathbf{x}, \mathbf{v}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ can be analogously decomposed.

$g_D(\mathbf{v}, z)$ and $g_B(\mathbf{v}, z)$ are discriminated against their disparate behaviors at $z \rightarrow \infty$ in the dissipation-less limit for real ω . To see this, we write $g_D(\mathbf{v}, z) = \sum_i g_D^{(i)}(\mathbf{v}, z)$, where $i = 1, 2, 3$ tracks e^{-kz} , $\cos(qz)$ and $\sin(qz)$, respectively. e^{-kz} does not depend on q and can be taken out of the integration over q in Eq. (7). Thus, $g_D^{(1)}(\mathbf{v}, z) \sim e^{-kz}$ and vanishes distant from the surface. The reason why $g_D^{(2)}(\mathbf{v}, z)$ and $g_D^{(3)}(\mathbf{v}, z)$ must also similarly vanish is because $\cos(qz)$ and $\sin(qz)$ are extremely oscillatory at $z \rightarrow \infty$, and hence upon integration over q , they make no contribution. One can reason that $g_D^{(i)}(\mathbf{v}, z) \sim e^{-kz}$ for large z , implying that $g_D(\mathbf{v}, z)$ decays away from the surface. On the contrary, $g_B(\mathbf{v}, z) \sim e^{i\tilde{\omega}z/v_z}$ is purely oscillatory without decay.

They are also discriminated by their dependences on the presence of surface. Upon taking away the surface, i.e. by replacing z with $z + z_0$ with $z_0 \rightarrow \infty$, $g_B(\mathbf{v}, z)$ would disappear invariably for any $\text{Im}(\omega + i/\tau) > 0$ while $g_D(\mathbf{v}, z)$ is almost not changed. This means that, $g_B(\mathbf{v}, z)$ represents genuine surface effects while $g_D(\mathbf{v}, z)$ is essentially a bulk property.

We call $g_D(\mathbf{v}, z)$ the diffusive component, for it roughly follows the profile of $\mathbf{E}(z)$, as if the electrons were driven by the latter. Actually, as to be shown in the next section, the electrical current density $\mathbf{J}_D(\mathbf{x}, t)$ arising from $g_D(\mathbf{v}, z)$ reduces to the ordinary Drude form¹.

We call $g_B(\mathbf{v}, z)$ the surface-ballistic component, because it would disappear without surface and ballistic motions. It features the phase factor $e^{i\omega_s z/v_z}$ with a simple physical interpretation: the phase $\varphi(z) = \frac{\omega_s z}{v_z}$ is accumulated when an electron leaves the surface and travels to the depth z without suffering a collision.

According to the above definition, all electrons moving toward the surface are sheerly diffusive and those bounced back are sheerly ballistic, while the electrons emerging from the surface are of a mixed character. Needless to say, the surface-ballistic component does not exist in bulk plasma waves even in the dissipation-less limit. It arises solely due to the presence of a surface.

In view of energy conversion, the foremost difference between $\mathbf{J}_D(\mathbf{x}, t)$ and $\mathbf{J}_B(\mathbf{x}, t)$ lies with their local orientations relative to $\mathbf{E}(\mathbf{x}, t)$. As expected from Drude's law, $\mathbf{J}_D(\mathbf{x}, t)$ is largely oriented normal to $\mathbf{E}(\mathbf{x}, t)$ and preserves the kinetic energy. In contrast, $\mathbf{J}_B(\mathbf{x}, t)$ will be shown to mostly align with $\mathbf{E}(\mathbf{x}, t)$ and does not preserve the kinetic energy. Actually, $\mathbf{J}_B(\mathbf{x}, t)$ diminishes it.

V. WORK DONE VIA DIFFUSIVE CURRENT

We proceed to calculate the power $P^{(1)}(t) = S\mathcal{P}^{(1)}(t)$ and $P^{(2)} = S\mathcal{P}^{(2)}(t)$ in Eq. (2). Here we have introduced $\mathcal{P}^{(1,2)}(t)$ to denote the work done per unit time per unit surface area. As shown in preceding section, the electrons can be organized in two portions, a diffusive portion and a surface-ballistic one. Hence, $\mathbf{J}(\mathbf{x}, t)$ and $\mathcal{P}^{(1,2)}(t)$ also naturally occurs in two portions, that is, $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_D(\mathbf{x}, t) + \mathbf{J}_B(\mathbf{x}, t)$ and $\mathcal{P}^{(1,2)}(t) = \mathcal{P}_D^{(1,2)}(t) + \mathcal{P}_B^{(1,2)}(t)$. In this section, we are concerned with the diffusive part. We show that $\mathcal{P}_D^{(1,2)}(t) \equiv 0$ for real ω in the dissipation-less limit, implying that diffusive currents do not destabilize the system.

To ease the discussions, we write $\mathbf{J}_{D/B}(\mathbf{x}, t) = (m/2\pi\hbar)^3 \int d^3\mathbf{v} \mathbf{J}_{D/B}(\mathbf{x}, t; v_x, v_z)$, where $\mathbf{J}_{D/B}(\mathbf{x}, t; v_x, v_z) = e\mathbf{v}g_{D/B}(\mathbf{x}, \mathbf{v}, t)$ stands for the contribution produced by a beam of electrons with velocity \mathbf{v} . The power can be written $\mathcal{P}_{D/B}^{(1,2)}(t) =$

$(m/2\pi\hbar)^3 \int_0^\infty dz \int d^3\mathbf{v} \mathcal{P}_{D/B}^{(1,2)}(v_x, v_z > 0, z, t)$, where

$$\left(\mathcal{P}_{D/B}^{(1)}(v_x, v_z > 0, z, t) \right) = \int \frac{d^2\mathbf{r}}{S} \left(\mathbf{J}_{D/B}(\mathbf{r}, z, t; v_x, v_z > 0) \cdot \mathbf{E}(\mathbf{r}, z, t) \right). \quad (11)$$

Here we have introduced

$$\mathbf{J}_{D/B}(\mathbf{r}, z, t; v_x, v_z > 0) = \Theta(v_z) \left[\mathbf{J}_{D/B}(\mathbf{r}, z, t; v_x, v_z) + \mathbf{J}_{D/B}(\mathbf{r}, z, t; v_x, -v_z) \right]. \quad (12)$$

Below we show that $\mathcal{P}_D^{(1,2)}(v_x, v_z > 0, z, t)$ vanishes, thereby proving that $\mathcal{P}_D^{(1,2)}(t)$ also vanishes.

We first establish the expression of $\mathbf{J}_D(\mathbf{x}, t; v_x, v_z > 0) = \text{Re} \left[\mathbf{J}_D(z, v_x, v_z > 0) e^{i(kx - \omega t)} \right]$. Let us begin with the x -component. By definition, we have

$$J_{D,x}(z, v_x, v_z > 0) = ev_x \left[g_{>,D}^{(1)}((v_x, v_y, v_z), z) + g_{<}(v_x, v_y, -v_z, z) \right],$$

where we have temporarily put $\mathbf{v} = (v_x, v_y, v_z)$. The coefficient of $\sin(qz)$ in $g_{>,D}^{(1)}((v_x, v_y, -v_z), z)$ exactly cancels out that in $g_{<}(\mathbf{x}, (v_x, v_y, -v_z), t)$. We find

$$J_{D,x}(z, v_x, v_z > 0) = \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \frac{2e^2}{v_z} \int \mathcal{D}q \rho_q \left(2 \cos(qz) \frac{qv_x v_z \frac{q}{k} + \tilde{\omega} \frac{v_x^2}{v_z}}{(\tilde{\omega}/v_z)^2 - q^2} + e^{-kz} \frac{kv_x v_z - \tilde{\omega} \frac{v_x^2}{v_z}}{(\tilde{\omega}/v_z)^2 + k^2} \right). \quad (13)$$

In the dissipation-less limit $\tilde{\omega}$ becomes real for real ω (i.e. assuming stable SPWs). $J_{D,x}(z, v_x, v_z > 0)$ also becomes real. As a result, $J_{D,x}(\mathbf{x}, t; v_x, v_z > 0) = J_{D,x}(z, v_x, v_z > 0) \cos(kx - \omega t)$.

The z -component can be similarly found. We have

$$J_{D,z}(z, v_x, v_z > 0) = ev_x \left[g_{>,D}^{(1)}((v_x, v_y, v_z), z) - g_{<}((v_x, v_y, -v_z), z) \right].$$

Now the coefficient of $\cos(qz)$ in $g_{>,D}^{(1)}((v_x, v_y, -v_z), z)$ exactly cancels out that in $g_{<}(\mathbf{x}, (v_x, v_y, -v_z), t)$. It yields

$$J_{D,z}(z, v_x, v_z > 0) = \left(\omega + \frac{i}{\tau} \right) \frac{v_z}{i} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \frac{2e^2}{v_z} \int \mathcal{D}q \rho_q \left(2 \sin(qz) \frac{q/k}{q^2 - (\tilde{\omega}/v_z)^2} + \frac{e^{-kz}}{k^2 + (\tilde{\omega}/v_z)^2} \right). \quad (14)$$

In the dissipation-less limit $\tilde{\omega}$ becomes real for real ω . Hence, $J_{D,z}(z, v_x, v_z > 0)$ becomes imaginary. Thus, we find $J_{D,z}(\mathbf{x}, t; v_x, v_z > 0) = iJ_{D,z}(z, v_x, v_z > 0) \sin(kx - \omega t)$. Summarizing, we get

$$\mathbf{J}_D(\mathbf{x}, t; v_x, v_z > 0) = \left(J_{D,x}(z, v_x, v_z > 0) \cos(kx - \omega t), iJ_{D,z}(z, v_x, v_z > 0) \sin(kx - \omega t) \right) \quad (15)$$

in the dissipation-less limit for real ω . Comparing this to the expression of $\mathbf{E}(\mathbf{x}, t)$ given in Eq. (3), we see that $\mathbf{J}_D(\mathbf{x}, t; v_x, v_z > 0)$ differs locally from $\mathbf{E}(\mathbf{x}, t)$ by $\pi/2$ in phase. Doing the integration over \mathbf{r} in Eq. (11), we immediately conclude that $\mathcal{P}_D^{(1,2)}(v_x, v_z > 0, z, t) \equiv 0$. Therefore, $\mathcal{P}_D^{(1,2)}(t) \equiv 0$, as stated in the beginning of this section.

As $\bar{\omega} = \omega + i/\tau$ is the single dominant frequency, we may retain only the leading terms in $\bar{\omega}$ in both the denominators and the numerators of the fractions involved in the integrands in Eqs. (13) and (14). For example, $\left[(\bar{\omega}/v_z)^2 - q^2 \right]^{-1} \approx v_z^2/\bar{\omega}^2$ (see Appendix). Under this approximation, we immediately recover the Drude result, namely, $\mathbf{J}_D(\mathbf{x}, t) = \text{Re} \left[\mathbf{J}_D(z) e^{i(kx - \omega t)} \right]$, with $\mathbf{J}_D(z) \approx \sigma_D \mathbf{E}(z)$, where $\sigma_D = (ne^2/m)(i/\bar{\omega})$ is no more than the Drude conductivity. This allows a straightforward estimate of $\mathcal{P}_D^{(1,2)}(t)$ for complex $\bar{\omega}$. Let us write $\bar{\omega} \approx \omega_s + i\Gamma$, where Γ can originate from either the instability or the dissipation or both, i.e. $\Gamma = \gamma + 1/\tau$. We find $\mathcal{P}_D^{(1)}(t) \approx (ne^2/m\omega_s)(\Gamma/\omega_s)S^{-1} \int d^2\mathbf{r} \int_0^\infty dz \mathbf{E}^2(\mathbf{x}, t) \approx \Gamma \frac{\pi \rho_s^2}{k} e^{2\gamma t}$ and similarly, $\mathcal{P}_D^{(2)}(t) \approx -\Gamma(\pi \rho_s^2/2k) e^{2\gamma t}$, where we have assumed Γ much smaller than ω_s and used $\omega_p^2 = 2\omega_s^2$. Then $\mathcal{P}_D^{(1)}(t) + \mathcal{P}_D^{(2)}(t) = \Gamma(\pi \rho_s^2/2k) e^{2\gamma t}$, which is positive and can be absorbed in $2\mathcal{E}_p(t)/\tau + \dot{\mathcal{E}}_p(t)$. Together with Eq. (5), the equation of energy balance (2) can be written

$$\gamma \approx -\frac{1}{\tau} - \frac{k}{\pi \rho_s^2} e^{-2\gamma t} \left(\mathcal{P}_B^{(1)}(t) + \mathcal{P}_B^{(2)}(t) \right). \quad (16)$$

Here the first term signifies dissipation and can be verified using the equation of motion in the Drude limit. $\mathcal{P}_D^{(2)}(t)$ is essential in obtaining the correct damping.

VI. WORK DONE VIA SURFACE-BALLISTIC CURRENT

Now we consider the work done via the surface-ballistic current, which are contributed only by electrons leaving the surface, i.e. $v_z > 0$. This is to be understood in the rest of this section. Hence, $\mathbf{J}_B(\mathbf{x}, t; v_x, -v_z) = 0$ in Eq. (12). Unlike its diffusive counterpart, $g_B(\mathbf{v}, z)$ depends on p . With Eqs. (8) and (9), it can be written $g_B(\mathbf{v}, z) = (1 + p) g_B^{(1)}(\mathbf{v}, z) - (1 - p) g_B^{(2)}(\mathbf{v}, z)$, where

$$\begin{pmatrix} g_B^{(1)}(\mathbf{v}, z) \\ g_B^{(2)}(\mathbf{v}, z) \end{pmatrix} = e^{i\frac{\tilde{\omega}z}{v_z}} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \frac{e}{v_z} \int_0^\infty \mathcal{D}q \rho_q \left(\frac{\frac{k(iv_x - v_z)}{(\tilde{\omega}/v_z)^2 + k^2} - \frac{2qv_z(q/k)}{(\tilde{\omega}/v_z)^2 - q^2}}{\frac{i(\tilde{\omega}/v_z)(iv_x - v_z)}{(\tilde{\omega}/v_z)^2 + k^2} + \frac{2\tilde{\omega}(v_x/v_z)}{(\tilde{\omega}/v_z)^2 - q^2}} \right). \quad (17)$$

We write $\tilde{\omega} = \Omega + i\Gamma$, with $\Omega = \omega_s - kv_x$ and $\Gamma = \gamma + 1/\tau$, and $Q = \Omega/v_z$. Up to the first order in Γ/ω_s , we may put

$$\begin{pmatrix} g_B^{(1)}(\mathbf{x}, \mathbf{v}, t) \\ g_B^{(2)}(\mathbf{x}, \mathbf{v}, t) \end{pmatrix} = e^{\gamma t - \frac{\Gamma z}{v_z}} \text{Re} \left\{ e^{i(kx + Qz - \omega_s t)} \left\{ \begin{pmatrix} g_c^{(1)}(\mathbf{v}) - ig_s^{(1)}(\mathbf{v}) \\ g_c^{(2)}(\mathbf{v}) - ig_s^{(2)}(\mathbf{v}) \end{pmatrix} + \frac{\Gamma}{\omega_s} \begin{pmatrix} -2g_s^{(1)}(\mathbf{v}) - 2ig_c^{(1)}(\mathbf{v}) \\ g_s^{(2)}(\mathbf{v}) + ig_c^{(2)}(\mathbf{v}) \end{pmatrix} \right\} \right\}, \quad (18)$$

where the components are given by $g_{c/s}^{(1,2)}(\mathbf{v}) = \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \frac{e}{v_z} \int_0^\infty \mathcal{D}q \rho_q g_{c/s}^{(1,2)}(\mathbf{v}, q)$ with

$$\begin{pmatrix} g_c^{(1)}(\mathbf{v}, q) & g_c^{(2)}(\mathbf{v}, q) \\ g_s^{(1)}(\mathbf{v}, q) & g_s^{(2)}(\mathbf{v}, q) \end{pmatrix} = \begin{pmatrix} -\frac{kv_z}{Q^2 + k^2} - \frac{2qv_z(q/k)}{Q^2 - q^2} & \frac{\Omega}{v_z} \left(\frac{2v_x}{Q^2 - q^2} - \frac{v_x}{Q^2 + k^2} \right) \\ -\frac{kv_x}{Q^2 + k^2} & \frac{v_z}{Q^2 + k^2} \frac{\Omega}{v_z} \end{pmatrix}. \quad (19)$$

In addition, we define

$$\begin{pmatrix} \mathbf{J}_{c/c/s}^{(1,2)}(\mathbf{x}, \mathbf{v}, t) \\ \mathbf{J}_{s,c/s}^{(1,2)}(\mathbf{x}, \mathbf{v}, t) \end{pmatrix} = e\mathbf{v}g_{c/s}^{(1,2)}(\mathbf{v}) e^{\gamma t - \frac{\Gamma z}{v_z}} \begin{pmatrix} \cos(kx + Qz - \omega_s t) \\ \sin(kx + Qz - \omega_s t) \end{pmatrix}. \quad (20)$$

Naturally we decompose

$$\mathbf{J}_B(\mathbf{x}, t; v_x, v_z) = (1 + p)\mathbf{J}_B^{(1)}(\mathbf{x}, \mathbf{v}, t) - (1 - p)\mathbf{J}_B^{(2)}(\mathbf{x}, \mathbf{v}, t), \quad (21)$$

where $\mathbf{J}_B^{(1,2)}(\mathbf{x}, \mathbf{v}, t) = e\mathbf{v}g_B^{(1,2)}(\mathbf{x}, \mathbf{v}, t)$. With Eq. (20), we have

$$\begin{pmatrix} \mathbf{J}_B^{(1)}(\mathbf{x}, \mathbf{v}, t) \\ \mathbf{J}_B^{(2)}(\mathbf{x}, \mathbf{v}, t) \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{c,c}^{(1)}(\mathbf{x}, \mathbf{v}, t) + \mathbf{J}_{s,s}^{(1)}(\mathbf{x}, \mathbf{v}, t) \\ \mathbf{J}_{c,c}^{(2)}(\mathbf{x}, \mathbf{v}, t) + \mathbf{J}_{s,s}^{(2)}(\mathbf{x}, \mathbf{v}, t) \end{pmatrix} + \frac{\Gamma}{\omega_s} \begin{pmatrix} 2\mathbf{J}_{s,c}^{(1)}(\mathbf{x}, \mathbf{v}, t) - 2\mathbf{J}_{c,s}^{(1)}(\mathbf{x}, \mathbf{v}, t) \\ -\mathbf{J}_{s,c}^{(2)}(\mathbf{x}, \mathbf{v}, t) + \mathbf{J}_{c,s}^{(2)}(\mathbf{x}, \mathbf{v}, t) \end{pmatrix}. \quad (22)$$

The power per unit surface area $\mathcal{P}_B^{(1,2)}(\mathbf{v}, z, t)$, introduced via Eq. (11), can be easily calculated. They are written

$$\mathcal{P}_B^{(1,2)}(\mathbf{v}, z, t) = \frac{e^2 \gamma t}{2} \left[(1 + p)\mathcal{P}_1^{(1,2)}(\mathbf{v}, z) - (1 - p)\mathcal{P}_2^{(1,2)}(\mathbf{v}, z) \right], \quad (23)$$

where, in terms of $J_{c/s,\mu}^{(1,2)}(\mathbf{v}) = \frac{1}{2}e v_\mu g_{c/s}^{(1,2)}(\mathbf{v})$, we have

$$\begin{pmatrix} \mathcal{P}_1^{(1)}(\mathbf{v}, z) \\ \mathcal{P}_2^{(1)}(\mathbf{v}, z) \end{pmatrix} = e^{-\frac{\Gamma z}{v_z}} E_x(z) \left[\begin{pmatrix} J_{s,x}^{(1)}(\mathbf{v}) + \frac{2\Gamma}{\omega_s} J_{c,x}^{(1)}(\mathbf{v}) \\ J_{s,x}^{(2)}(\mathbf{v}) - \frac{\Gamma}{\omega_s} J_{c,x}^{(2)}(\mathbf{v}) \end{pmatrix} \cos(Qz) - \begin{pmatrix} J_{c,x}^{(1)}(\mathbf{v}) - \frac{2\Gamma}{\omega_s} J_{s,x}^{(1)}(\mathbf{v}) \\ J_{c,x}^{(2)}(\mathbf{v}) + \frac{\Gamma}{\omega_s} J_{s,x}^{(2)}(\mathbf{v}) \end{pmatrix} \sin(Qz) \right] \\ + e^{-\frac{\Gamma z}{v_z}} E_z(z) \left[\begin{pmatrix} J_{s,z}^{(1)}(\mathbf{v}) + \frac{2\Gamma}{\omega_s} J_{c,z}^{(1)}(\mathbf{v}) \\ J_{s,z}^{(2)}(\mathbf{v}) - \frac{\Gamma}{\omega_s} J_{c,z}^{(2)}(\mathbf{v}) \end{pmatrix} \sin(Qz) + \begin{pmatrix} J_{c,z}^{(1)}(\mathbf{v}) - \frac{2\Gamma}{\omega_s} J_{s,z}^{(1)}(\mathbf{v}) \\ J_{c,z}^{(2)}(\mathbf{v}) + \frac{\Gamma}{\omega_s} J_{s,z}^{(2)}(\mathbf{v}) \end{pmatrix} \cos(Qz) \right], \quad (24)$$

and

$$\begin{pmatrix} \mathcal{P}_1^{(2)}(\mathbf{v}, z) \\ \mathcal{P}_2^{(2)}(\mathbf{v}, z) \end{pmatrix} = E_x(z) \begin{pmatrix} J_{s,x}^{(1)}(\mathbf{v}) + \frac{2\Gamma}{\omega_s} J_{c,x}^{(1)}(\mathbf{v}) \\ J_{s,x}^{(2)}(\mathbf{v}) - \frac{\Gamma}{\omega_s} J_{c,x}^{(2)}(\mathbf{v}) \end{pmatrix} + E_z(z) \begin{pmatrix} J_{c,z}^{(1)}(\mathbf{v}) - \frac{2\Gamma}{\omega_s} J_{s,z}^{(1)}(\mathbf{v}) \\ J_{c,z}^{(2)}(\mathbf{v}) + \frac{\Gamma}{\omega_s} J_{s,z}^{(2)}(\mathbf{v}) \end{pmatrix}. \quad (25)$$

The oscillatory factors, $\cos(Qz)$ and $\sin(Qz)$, which appear in $\mathcal{P}_{1,2}^{(1)}(\mathbf{v}, z)$, do not appear in $\mathcal{P}_{1,2}^{(2)}(\mathbf{v}, z)$, for the latter involves only the current density on the surface, i.e. $J_z(\mathbf{r}, 0, t)$, as seen in Eq. (2). Upon the integration over z , this fact makes $\mathcal{P}_{1,2}^{(1)}(\mathbf{v}, z)$ a higher order contribution in $\kappa = k/k_s$, in comparison with $\mathcal{P}_{1,2}^{(2)}(\mathbf{v}, z)$, to the total power.

So far, except for those inherent in Boltzmann's theory, we have not made any other approximations in obtaining the above expressions. To make progress, let us take $E_x(z) \approx E_z(z) \approx E(z) = 2\pi\rho_s e^{-kz}$, as in Sec. II, with which it is easy to show that $\mathcal{P}_B^{(1)}(t)$ is negligible in comparison with $\mathcal{P}_B^{(2)}(t)$. Writing $\mathcal{P}_{1,2}^{(1,2)}(\mathbf{v}) = \int_0^\infty dz \mathcal{P}_{1,2}^{(1,2)}(\mathbf{v}, z)$ and doing the integration, we arrive at

$$\begin{pmatrix} \mathcal{P}_1^{(1)}(\mathbf{v}) \\ \mathcal{P}_2^{(1)}(\mathbf{v}) \end{pmatrix} = \frac{2\pi\rho_s \left(k + \frac{\Gamma}{v_z}\right)}{\left(k + \frac{\Gamma}{v_z}\right)^2 + Q^2} \begin{pmatrix} J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v}) + \frac{2\Gamma}{\omega_s} \left(J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v})\right) \\ J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v}) - \frac{\Gamma}{\omega_s} \left(J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v})\right) \end{pmatrix} - \frac{2\pi\rho_s Q}{\left(k + \frac{\Gamma}{v_z}\right)^2 + Q^2} \begin{pmatrix} J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v}) - \frac{2\Gamma}{\omega_s} \left(J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v})\right) \\ J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v}) + \frac{\Gamma}{\omega_s} \left(J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v})\right) \end{pmatrix}, \quad (26)$$

which, after being multiplied by k , vanishes for $k = 0$. By contrast,

$$\begin{pmatrix} \mathcal{P}_1^{(2)}(\mathbf{v}) \\ \mathcal{P}_2^{(2)}(\mathbf{v}) \end{pmatrix} = \frac{2\pi\rho_s}{k} \begin{pmatrix} J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v}) + \frac{2\Gamma}{\omega_s} \left(J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v})\right) \\ J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v}) - \frac{\Gamma}{\omega_s} \left(J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v})\right) \end{pmatrix} \quad (27)$$

remains finite under the same circumstances, confirming that $\mathcal{P}_B^{(2)}(t)$ dominates over $\mathcal{P}_B^{(1)}(t)$ for small k . Indeed, as shown in the Appendix, $\mathcal{P}_B^{(1)}(t) \sim \kappa^2 \mathcal{P}_B^{(2)}(t)$. Hereafter we shall focus on $\mathcal{P}_B^{(2)}(t)$ and defer the discussions of $\mathcal{P}_B^{(1)}(t)$ to the Appendix.

Let us write

$$\mathcal{P}_B^{(2)}(t) = e^{2\gamma t} \frac{\pi\rho_s^2}{k} \left(b \frac{\Gamma}{\omega_s} - a\right),$$

where the coefficients are given by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left(\frac{m}{2\pi\hbar}\right)^3 \int \frac{d^3\mathbf{v}}{\rho_s} \begin{pmatrix} (1-p) \left(J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v})\right) - (1+p) \left(J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v})\right) \\ (1-p) \left(J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v})\right) + 2(1+p) \left(J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v})\right) \end{pmatrix}. \quad (28)$$

Here the integration is restricted to $v_z \geq 0$. Up to the linear order in k , we find

$$\int \frac{d^3\mathbf{v}}{\rho_s} \left(J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v})\right) = - \int \mathcal{D}q \frac{\rho_q}{\rho_s} \int d^3\mathbf{v} \left(-\frac{\partial f_0}{\partial \varepsilon}\right) \frac{e^2}{2v_z} \left(\frac{k(v_x^2 + v_z^2)}{(\omega_s/v_z)^2} + \frac{2q^2 v_z^2/k}{(\omega_s/v_z)^2 - q^2}\right) < 0, \quad (29)$$

which is obviously negative. Additionally,

$$\int \frac{d^3\mathbf{v}}{\rho_s} \left(J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v})\right) = \int \mathcal{D}q \frac{\rho_q}{\rho_s} \int d^3\mathbf{v} \left(-\frac{\partial f_0}{\partial \varepsilon}\right) \frac{e^2}{v_z} \frac{kv_x^2}{(\omega_s/v_z)^2 - q^2} \frac{(\omega_s/v_z)^2 + q^2}{(\omega_s/v_z)^2 - q^2} > 0, \quad (30)$$

which is obviously positive. It follows that $a > 0$ inevitably and hence $\mathcal{P}_B^{(2)}(t) < 0$. In the Appendix, we show that $b \approx (c_2\kappa^2 - c_1\kappa)\omega_s$, where $c_{1,2} > 0$ are constants much smaller than unity. Thus, b will be subsequently left out. Inclusion of b leads to bigger γ_0 . Substituting $\mathcal{P}_B^{(2)}(t)$ in the energy balance equation (16) yields $\gamma \approx \gamma_0 - \frac{1}{\tau}$, which can be positive for big τ . Here $\gamma_0 = a$. The last term in the integrand in Eq. (29) makes up the prevailing contribution to γ_0 .

VII. INSTABILITY AND AMPLIFICATION

We have demonstrated that the electrical current have two disparate components, which we call diffusive and surface-ballistic, respectively. The separation is not based on whether electronic collisions are frequent or not. An inherent feature distinguishes them. The diffusive current obeys the conventional Drude's law, according to which the energy stored in the electric field could only be converted into the kinetic energy of the electrons and eventually dissipated away. The surface-ballistic current, however, cannot be described by the Drude's law. Unlike the diffusive component, the surface-ballistic current converts the kinetic energy into the electrostatic potential energy, thereby destabilizing the system.

The energy balance equation, derived irrespective of the particulars of the underlying dynamics and the surface properties, requires that the instability is accompanied by an intrinsic amplification of SPWs. In the dissipation-less limit, the amplification rate γ simply amounts to a given in Eq. (28). For convenience, let us write $a = (1+p)a_1 + (1-p)a_2$, where a_1 and a_2 arise from Eqs. (29) and (30), respectively. Performing the integral in Eq. (29), we arrive at

$$a_1 = \frac{3\omega_s}{16\pi} \kappa \int_0^{q_c} \mathcal{D}q \frac{\rho_q}{\rho_s} \left(\frac{3}{4} + \left(\frac{q}{k}\right)^2 \int_0^1 dr \frac{4r^3}{1 - (q/k)^2 \kappa^2 r^2}\right).$$

Similarly, we have

$$a_2 = \frac{3\omega_s}{8\pi} \kappa \int_0^{q_c} \mathcal{D}q \frac{\rho_q}{\rho_s} \int_0^1 dr \frac{(1-r^2)r}{1 - (q/k)^2 \kappa^2 r^2} \frac{1 - (q/k)^2 \kappa^2 r^2}{1 + (q/k)^2 \kappa^2 r^2}.$$

With $\rho_q \approx \rho_s$ and $q_c \approx k_s$, the integrals can be analytically calculated. Up to the linear order in κ , we obtain

$$\gamma_0 \approx \omega_s \left(\frac{3(1+p)}{4\pi} - \frac{3(3p-1)}{32} \kappa \right), \quad (31)$$

in agreement with our previous work²⁴. The constant term in γ_0 stems from the second term in Eq. (29). The equation shows that the system is more unstable with specular surfaces than diffuse ones.

VIII. DISCUSSIONS AND SUMMARY

We have presented a systematic analysis of the energy conversion in ballistic SPWs. We find that ballistic motions could destabilize the system when a surface is present and lend SPWs an intrinsic amplification channel with a rate γ_0 given by Eq. (31). Via this channel, SPWs can extract energy from the Fermi sea and amplify themselves provided the loss channels are sufficiently suppressed. In the present calculations, we have explicitly shown that the losses due to electronic collisions (Joule heat) simply subtract the amplification rate by $1/\tau$. The framework set up in Sec. II, however, is generic and can also be used to deal with other losses. In what follows, we discuss some of them.

First of all, SPWs can lose energy to individual electronic motions by means of Landau damping³⁴. This will happen only if ω_s crosses the spectrum of these motions. At zero temperature, this means $\omega_s \sim qv_F$, which would not be possible if $q_c \sim k_s$. To account for the possibility, we can let q_c slightly exceed k_s and then put $\frac{1}{\omega_s^2 - q^2 v_z^2} = \left[\frac{1}{\omega_s^2 - q^2 v_z^2} \right] - \frac{i}{2\omega_s} \delta(\omega_s - qv_z)$, where the square bracket takes the Cauchy principal value. The imaginary part out of this factor can bring about addi-

tional losses giving rise to Landau damping, which, however, is suppressed due to the tiny volume of q available for the process. The damping rate is higher order in κ and can be neglected²⁴.

In addition, inter-band transitions can also absorb SPW energy. There are two aspects of such transitions. First, they might modify the frequency of SPWs. The modified frequency can be determined using the equation of motion method, as described in our previous work²⁴. Second, they generate extra electrical currents, which must be added up to the currents obtained using Boltzmann's theory (Sec. III). The total currents are then used in the energy balance equation. Losses are then incurred for the extra currents. If ω_s stays far off the transition threshold, however, such losses are insignificant. A detailed study along this line will be published in the future.

Finally, SPWs can be converted into radiation on rough or structured surfaces¹⁰. This process is largely refrained by the wave number mismatch between SPWs and the radiation at the same frequency. The emanated radiation is not necessarily unwanted. For example, it can serve as a terahertz radiation source³⁵. Through this process, the energy stored in the Fermi sea may be released.

In summary, we have derived a generic formalism for studying energy conversion processes in systems with boundaries. We have applied it to SPWs and shown that the Fermi sea of metals is unstable thanks to the interplay between ballistic electronic motions and the presence of a surface. We hope this work will stimulate more interest in ballistic SPWs from both the theoretical and experimental communities.

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Appendix: Calculation of b and $\mathcal{P}_B^{(1)}(t)$

In this appendix, we provide some details of the calculations which were skipped in the main text. Without affecting the conclusions we presume $\rho_q \approx \rho_s$ and $q_c \approx k_s$ to obtain analytical expressions. We first show that $b \approx \omega_s [c_1 \kappa - c_2 \kappa^2]$, where $\kappa = k/k_s$. As suggested by Eq. (28), we may put $b = (1+p)b_1 + (1-p)b_2$. We then have

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \left(\frac{m}{2\pi\hbar} \right)^3 \int \frac{d^3\mathbf{v}}{\rho_s} \left(2 \begin{pmatrix} J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v}) \\ J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v}) \end{pmatrix} \right). \quad (A.1)$$

Using the definition of $J_{c/s}^{(1,2)}(\mathbf{v})$, we get

$$\int \frac{d^3\mathbf{v}}{\rho_s} (J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v})) = - \int \mathcal{D}q \frac{\rho_q}{\rho_s} \int d^3\mathbf{v} \left(-\frac{\partial f_0}{\partial \epsilon} \right) \frac{e^2 (q^2/k) v_x v_z}{v_z (Q^2 - q^2)}. \quad (A.2)$$

We expand $\frac{1}{Q^2 - q^2} = \frac{1}{(\omega_s/v_z)^2 - q^2} \left(1 + \frac{2\omega_s k v_x}{\omega_s^2 - q^2 v_z^2} \right)$. The first term does not contribute to the integration, because the resulting integrand is odd in v_x , and upon integration it vanishes. So we get

$$\int \frac{d^3\mathbf{v}}{\rho_s} (J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v})) = -2e^2 \int \mathcal{D}q \frac{\rho_q}{\rho_s} \int d^3\mathbf{v} \left(-\frac{\partial f_0}{\partial \epsilon} \right) \frac{q^2 v_x^2 v_z^2}{\omega_s^2 - q^2 v_z^2} \frac{\omega_s}{\omega_s^2 - q^2 v_z^2}. \quad (A.3)$$

Let us expand $\frac{1}{\omega_s^2 - q^2 v_z^2} = \frac{1}{\omega_s^2} \sum_{N=0}^{\infty} \left(\frac{qv_z}{\omega_s}\right)^{2N}$. One can show that the N -th term in this series comprises a contribution of the order N^{-2} to Eq. (A.3). We retain only the leading term. Performing the integration, we arrive at

$$b_1 \approx \frac{2\omega_s}{5\pi} \left(\frac{\pi}{2}\kappa^2 - \kappa\right). \quad (\text{A.4})$$

Handling b_2 similarly, we get

$$b_2 \approx \frac{2\omega_s}{5\pi} \left(\kappa - \frac{\pi}{4}\kappa^2\right). \quad (\text{A.5})$$

Putting together, we find

$$b = \frac{\omega_s}{10} \left[(1 + 3p)\kappa^2 - (8p/\pi)\kappa\right]. \quad (\text{A.6})$$

Note that $\frac{|b|}{\omega_s}$ hardly exceed $\frac{2}{5\pi^2} \approx 0.04$ for any p in the range $0 < \kappa < 1$. This proves our statement.

Now we look at $\mathcal{P}_B^{(1)}(t)$, which is written

$$\mathcal{P}_B^{(1)}(t) = e^{2\gamma t} \frac{\pi\rho_s^2}{k} \left(b^{(1)} \frac{\Gamma}{\omega_s} - a^{(1)}\right). \quad (\text{A.7})$$

In the rest of this section we calculate $b^{(1)}$ and $a^{(1)}$. In the first place, we have

$$\mathcal{P}_1^{(1)}(\mathbf{v}) = \frac{2\pi\rho_s}{\bar{k}^2 + Q^2} \left[k \left(J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v}) \right) - Q \left(J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v}) \right) \right] + \frac{\Gamma}{\omega_s} \left(3Q \left(J_{s,x}^{(1)}(\mathbf{v}) + J_{c,z}^{(1)}(\mathbf{v}) \right) + 2k \left(J_{c,x}^{(1)}(\mathbf{v}) - J_{s,z}^{(1)}(\mathbf{v}) \right) \right). \quad (\text{A.8})$$

Here $\bar{k} = k + \Gamma/v_z$, which can be replaced with k without affecting the results. Additionally,

$$\mathcal{P}_2^{(1)}(\mathbf{v}) = \frac{2\pi\rho_s}{\bar{k}^2 + Q^2} \left(k \left(J_{s,x}^{(2)}(\mathbf{v}) + J_{c,z}^{(2)}(\mathbf{v}) \right) - Q \left(J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v}) \right) \right) - \frac{\Gamma}{\omega_s} \frac{2\pi\rho_s k}{\bar{k}^2 + Q^2} \left(J_{c,x}^{(2)}(\mathbf{v}) - J_{s,z}^{(2)}(\mathbf{v}) \right). \quad (\text{A.9})$$

Using the same tricks in calculating b , we easily get

$$b^{(1)} \approx \frac{3\omega_s}{160} \left(\kappa^2 - \frac{12}{\pi}\kappa\right) (1 + p). \quad (\text{A.10})$$

Now $a^{(1)}$ can be split $a^{(1)} = (1 + p)a_1^{(1)} - (1 - p)a_2^{(1)}$. We obtain

$$a_1^{(1)} \approx \frac{5\omega_s}{32}\kappa^3, \quad a_2^{(1)} \approx \frac{\omega_s}{8\pi} \left(\kappa^2 - \frac{\pi}{2}\kappa^3\right). \quad (\text{A.11})$$

We then see that $\mathcal{P}_B^{(1)}(t)$ is negligibly small in comparison with $\mathcal{P}_B^{(2)}(t)$.

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¹ N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976).

² C. J. Pethick and D. G. Ravenhall, *Ann. Phys.* 183, 131 (1988).

³ P. Philip, *Advanced Solid State Physics* (Cambridge University Press, 2012).

⁴ T. W. Ebbesen, C. Genet and S. I. Bozhevolnyi, *Phys. Today* 61, 44 (2008).

⁵ E. Ozbay, *Science* 311, 189 (2006).

⁶ A. V. Zayats, I. S. Igor and A. A. Maradudin, *Phys. Rep.* 408, 131 (2005).

⁷ W. L. Barnes, A. Dereux and T. W. Ebbesen, *Nature* 424, 824

(2003).

⁸ R. H. Ritchie, *Phys. Rev.* 106, 874 (1957).

⁹ R. A. Ferrell, *Phys. Rev.* 111, 1214 (1958).

¹⁰ H. Raether, *Surface plasmons on smooth and rough surfaces and on gratings* (Springer Berlin Heidelberg, 1988).

¹¹ B. Rothenhäusler and K. Wolfgang, *Nature* 332, 615 (1988).

¹² S. A. Maier, *Plasmonics: fundamentals and applications* (Springer Science & Business Media, 2007).

¹³ J. M. Pitarke, V. M. Silkin, E. V. Chulkov and P. M. Echenique, *Rep. Prog. Phys.* 70, 1 (2007).

¹⁴ M. L. Brongersma and P. G. Kik, *Surface plasmon nanophotonics* (Springer, 2007).

- ¹⁵ P. Berini and I. De Leon, Nat. Photonics 6, 16 (2012).
- ¹⁶ D. Y. Fedyanin and A. Y. Arsenin, Opt. Express 19, 12524 (2011).
- ¹⁷ D. Y. Fedyanin, A. V. Arsenin and A. V. Zayats, Nano Lett. 12, 2459 (2012).
- ¹⁸ I. De Leon and P. Berini, Nat. Photonics 4, 382 (2010); Phys. Rev. B 78, R161401 (2008).
- ¹⁹ J. Seidel, S. Frafstrom and L. Eng, Phys. Rev. Lett. 94, 177401 (2005).
- ²⁰ D. J. Bergman and M. I. Stockman, Phys. Rev. Lett. 90, 027402 (2003).
- ²¹ W. L. Barnes, J. of Optics A 8, S87 (2006).
- ²² F. Javier Garca de Abajo *et al.*, Faraday Discussions 178, 123 (2015).
- ²³ P. R. West, S. Ishii, G. V. Naik, N. K. Emani, V. M. Shalaev and A. Boltasseva, Laser & Photonics Reviews 4, 795 (2010).
- ²⁴ H.-Y. Deng, K. Wakabayashi and C.-H. Lam, arXiv: 1701.01060 (2017).
- ²⁵ J. Dawson, *Phys. Fluids* 4, 869 (1961).
- ²⁶ This equation can also be derived from Boltzmann's equation, $(\partial_t + \tau^{-1} + \mathbf{v} \cdot \partial_{\mathbf{x}})g(\mathbf{x}, \mathbf{v}, t) + (\mathbf{F}/m) \cdot \partial_{\mathbf{v}}f_0(\mathbf{v}) = -(\mathbf{F}/m) \cdot \partial_{\mathbf{v}}g(\mathbf{x}, \mathbf{v}, t)$, where f_0 and g are the equilibrium and non-equilibrium part of the Boltzmann distribution function, respectively, m is the electron mass and \mathbf{F} is the total force (excluding the part taken care of by the τ^{-1} term) acting on the electrons. We may write $\mathbf{F} = e\mathbf{E} + \mathbf{F}_s$, where e is the electron charge and \mathbf{F}_s is the force that prevents the electrons from escaping the metal. We may write $\mathbf{F}_s = e\mathbf{E}_s = -e\partial_{\mathbf{x}}\phi_s$, with ϕ_s denotes the surface potential. For an ideal surface \mathbf{F}_s should vanish everywhere except on the surface and point normal to the surface. Keeping g to the first order in \mathbf{E} , we can write $(\mathbf{F}/m) \cdot \partial_{\mathbf{v}}g = (\mathbf{F}_s/m) \cdot \partial_{\mathbf{v}}g$. Now multiplying the equation by e and integrating it over \mathbf{v} , we find $(\partial_t + \tau^{-1})\rho + \partial_{\mathbf{x}} \cdot \mathbf{J} = -e \int d\mathbf{v}(\mathbf{F}_s/m) \cdot \partial_{\mathbf{v}}g$. We cannot proceed further without knowing \mathbf{F}_s , whose details are generally difficult to know and could vary greatly from one sample to another. Despite this, we can fix it by demanding that the equation of continuity holds, i.e. $e \int d\mathbf{v}(\mathbf{F}_s/m) \cdot \partial_{\mathbf{v}}g = \delta(z)J_z(\mathbf{x}, t)$. This then leads to equation (1). This derivation makes it clear that this term can be traced back to the force exerted by the surface. From Boltzmann's equation, one can show that the total energy $E_k + E_s + E_p$ is conserved for $\tau \rightarrow \infty$, where $E_k(t) = (m/2\pi\hbar)^3 \int d\mathbf{x} \int d\mathbf{v}(m/2)\mathbf{v}^2 (f_0(\mathbf{v}) + g(\mathbf{x}, \mathbf{v}, t))$, $E_s(t) = \int d\mathbf{x}\phi_s(\mathbf{x})\rho(\mathbf{x}, t)$ and $E_p(t) = \int d\mathbf{x}\phi(\mathbf{x}, t)\rho(\mathbf{x}, t)$.
- ²⁷ On using $\hbar\omega_s = \sqrt{\frac{2\pi n e^2 \hbar}{m}}$ and $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$ as well as $k_F^3 = 6\pi^2 n$, we obtain $\frac{\hbar\omega_s}{\varepsilon_F} = \frac{\sqrt{8\pi}}{(6\pi^2)^{2/3}} \left(\frac{n_0}{n}\right)^{1/6} \approx 0.5 \left(\frac{n_0}{n}\right)^{1/6}$, where $n_0 = \left(\frac{e\sqrt{m}}{\hbar}\right)^6 \approx 10^{24}\text{cm}^{-3}$ comparable to n in magnitude. Thus, $\hbar\omega_s \sim \varepsilon_F$ and $k_s \sim k_F \sim n^{1/3}$. As such, we may put $q_c \sim k_s$.
- ²⁸ We have $\mathcal{E}_p(t) = (1/2) \int \frac{d\mathbf{x}}{S} \int_0^z \rho(\mathbf{x}, t)\phi(\mathbf{x}, t)$. Let $\rho(\mathbf{x}, t) \approx \rho_s e^{\gamma t} \cos(kx - \omega_s t)\delta(z)$. Hence $\phi(\mathbf{x}, t) \approx (2\pi\rho_s/k)e^{\gamma t - kz} \cos(kx - \omega_s t)$. Thus, $\mathcal{E}_p(t) = \frac{\pi\rho_s^2}{2k} \int_0^\infty dz\delta(z)e^{-kz} = \frac{\pi\rho_s^2}{2k} \frac{1}{2}$. Here the integration gives 1/2, because the surface charges actually peak at $z = 0$ and therefore only correspond to half of the Dirac function. This becomes evident if we replace $\delta(z)$ with e.g. $\frac{1}{\pi} \frac{\eta}{\eta^2 + z^2}$, where η is a positive infinitesimal.
- ²⁹ K. Fuchs, Proc. Camb. Phil. Soc. 34, 100 (1938).
- ³⁰ G. E. H. Reuter and E. H. Sondheimer, Proc. R. Soc. Lond. A 195, 338 (1948).
- ³¹ J. M. Ziman, *Electrons and Phonons: the theory of transport phenomena in solids* (Oxford University Press, 2001).
- ³² M. I. Kaganov, G. Y. Lyubarskiy and A. G. Mitina, Phys. Rep. 288, 291 (1997).
- ³³ G. P. Zhigal'skii and B. K. Jones, *The physical properties of thin metal films* (Taylor & Francis, London, 2003).
- ³⁴ L. Landau, J. Phys. USSR X, 25 (1946).
- ³⁵ D. K. Polyushkin, E. Hendry and W. L. Barnes, Appl. Phys. B 120, 53 (2015).